

# RIEMANNIAN CURVATURE OF THE NONCOMMUTATIVE 3-SPHERE

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**ABSTRACT.** In order to investigate to what extent the calculus of classical (pseudo-)Riemannian manifolds can be extended to a noncommutative setting, we introduce pseudo-Riemannian calculi of modules over noncommutative algebras. In this framework, it is possible to prove an analogue of Levi-Civita's theorem, stating that there exists at most one torsion-free and metric connection for a given (metric) module, satisfying the requirements of a real metric calculus. Furthermore, the corresponding curvature operator has the same symmetry properties as the classical Riemannian curvature. As our main motivating example, we consider a pseudo-Riemannian calculus over the noncommutative 3-sphere and explicitly determine the torsion-free and metric connection, as well as the curvature operator together with its scalar curvature.

## 1. INTRODUCTION

The topological aspects of noncommutative geometry have been extensively developed over the last decades, and there is a fine understanding of how geometrical concepts generalize (or not) to the noncommutative setting. Moreover, via spectral triples and Dirac operators, metric aspects have also been thoroughly studied [Con94]. In particular, a noncommutative connection and curvature formalism is worked out by A. Connes in [Con80], and now there is also an understanding of scalar curvature in terms of heat kernel expansions for spectral triples.

In recent years, several authors have made progress in computing the scalar curvature for noncommutative tori, defined as a particular term in the asymptotic heat kernel expansion, in analogy with classical Riemannian geometry [CM14, FK15, FK13, LM15]. The novelty of this formulation is that it can be used to prove index type theorems, such as a noncommutative Gauss-Bonnet theorem, which can be shown to hold for conformal perturbations of the flat metric for the noncommutative torus [FK12, CT11]. The computations rely on pseudo-differential calculus and are highly technical and analytical in nature. These results certainly have a profound impact in the field of noncommutative geometry. An interesting question is the existence of a curvature tensor, whose scalar curvature coincides with the one arising from the heat kernel expansion.

In an interesting paper by J. Rosenberg [Ros13], an algebraic approach to curvature of the noncommutative torus is taken, much in the spirit of noncommutative differential geometry [Con80]. It turns out to be possible to construct a Levi-Civita connection for certain classes of noncommutative tori, whose curvature tensor and scalar curvature can easily be computed.

In this paper, we try to understand some of the prerequisites for introducing traditional Riemannian geometry over a noncommutative algebra, and formalize these ideas in *pseudo-Riemannian calculi* for which several classical results hold;

in particular, there exists at most one torsion-free and metric connection. Furthermore, under certain hermiticity assumptions, the curvature tensor has all the symmetries one finds in the differential geometric setting. Although our framework is admittedly quite restrictive, and only a few noncommutative manifolds fulfill the requirements, we believe that our results contribute to the understanding of noncommutative Riemannian geometry, by studying particular and well known examples: the noncommutative torus and, foremost, the noncommutative 3-sphere [Mat91a]. In these examples, there are natural choices of modules, corresponding to (sections of) tangent bundles, which presents themselves when considering the manifolds as embedded in Euclidean space (see [Arn14, AH14, AHH12, ABH<sup>+</sup>09] for similar approaches making use of embeddings in Euclidean space). Let us also point out that there are several other related approaches to Riemannian structures in noncommutative geometry; see e.g. [CFF93, DVMMM96, AC10, BM11].

For our main example, the noncommutative 3-sphere, we find it interesting that our computations seem to introduce a type of noncommutative local tangent bundle, in the following sense (as discussed in Section 6.3). In general, the module of vector fields is not a free module, which impedes to work with directly from a computational point of view. Therefore, one usually considers the manifold chart by chart and carry out pointwise calculations. In noncommutative geometry, points are generally not accessible, but the fact that the tangent bundle is locally free is useful. However, since the objects we work with are intrinsically global, the restriction of vector fields to a chart in the noncommutative setting is not immediate. Instead, we extend a local basis of vector fields to global vector fields and consider the set of vector fields in the local chart as a free submodule of the tangent bundle, generated by the globalized vector fields. From a classical point of view, computations may equally well be done with these vector fields, keeping in mind that results can only be trusted for points that belong to the given chart. Furthermore, we introduce an Ore localization of the noncommutative 3-sphere, which is in direct analogy with the algebra of functions in the chart provided by the classical Hopf coordinates.

The paper is organized as follows: In Section 2 we introduce a few basic concepts of noncommutative algebra that will be used throughout the paper, in order to fix our notation. Section 3 introduces pseudo-Riemannian calculi, which provides a computational framework for Riemannian geometry over noncommutative algebras. In Section 4 the symmetry properties of the curvature of a pseudo-Riemannian calculus is discussed, as well as the possibility of introducing a scalar curvature. Section 5 presents the noncommutative torus in the framework of pseudo-Riemannian calculi, and we show that a unique (flat) torsion-free and metric connection exists. In Section 6 we introduce the main motivating example for this paper, the noncommutative 3-sphere. A real pseudo-Riemannian calculus is constructed, including the unique torsion-free and metric connection and, furthermore, the scalar curvature is computed. Finally, in Section 6.3, we discuss aspects of noncommutative localization in the context of the noncommutative 3-sphere.

## 2. PRELIMINARIES

In this section we shall recall the definitions of a few basic algebraic objects, in order to set the notation for the rest of the paper. In the following,  $\mathcal{A}$  will denote a unital  $*$ -algebra (over  $\mathbb{C}$ ) with center  $\mathcal{Z}(\mathcal{A})$ . The set of derivations of  $\mathcal{A}$  (into

$\mathcal{A}$ ) is denoted by  $\text{Der}(\mathcal{A})$ , and for any derivation  $d \in \text{Der}(\mathcal{A})$ , there is a hermitian conjugate  $d^*$ , given by  $d^*(a) = (d(a^*))^*$ ; a derivation is called hermitian if  $d^* = d$ .

In this paper we shall mainly be concerned with right  $\mathcal{A}$ -modules. In particular, the free (right)  $\mathcal{A}$ -module  $(\mathcal{A})^n$  has a canonical basis given by  $\{e_1, \dots, e_n\}$  where

$$e_i = (0, \dots, 0, \mathbb{1}, 0, \dots, 0)$$

with the only nonzero element in the  $i$ 'th position. An element  $U \in (\mathcal{A})^n$  can be written as  $U = e_i U^i$  (with an implicit sum over  $i$  from 1 to  $n$ ) for some (uniquely determined) elements  $U^1, \dots, U^n \in \mathcal{A}$ .

**Definition 2.1.** Let  $M$  be a right  $\mathcal{A}$ -module. A map  $h : M \times M \rightarrow \mathcal{A}$  is called a hermitian form on  $M$  if

$$\begin{aligned} h(U, V + W) &= h(U, V) + h(U, W) \\ h(U, Va) &= h(U, V)a \\ h(U, V)^* &= h(V, U). \end{aligned}$$

A hermitian form is *non-degenerate* if  $h(U, V) = 0$  for all  $V \in M$  implies that  $U = 0$ . For brevity, we simply refer to a non-degenerate hermitian form as a *metric* on  $M$ . The pair  $(M, h)$ , where  $M$  is a right  $\mathcal{A}$ -module and  $h$  is a hermitian form on  $M$ , is called a *(right) hermitian  $\mathcal{A}$ -module*. If  $h$  is a metric, we say that  $(M, h)$  is a *(right) metric  $\mathcal{A}$ -module*.

Let us introduce affine connections on a right  $\mathcal{A}$ -module, adjusted to fit the particular setting of this paper.

**Definition 2.2.** Let  $M$  be a right  $\mathcal{A}$ -module and let  $\mathfrak{g} \subseteq \text{Der}(\mathcal{A})$  be a (real) Lie algebra of hermitian derivations. An affine connection on  $(M, \mathfrak{g})$  is a map  $\nabla : \mathfrak{g} \times M \rightarrow M$  such that

- (1)  $\nabla_d(U + V) = \nabla_d U + \nabla_d V$ ,
- (2)  $\nabla_{\lambda d + d'} U = \lambda \nabla_d U + \nabla_{d'} U$ ,
- (3)  $\nabla_d(Ua) = (\nabla_d U)a + U d(a)$ ,

for all  $U, V \in M$ ,  $d, d' \in \mathfrak{g}$ ,  $a \in \mathcal{A}$  and  $\lambda \in \mathbb{R}$ .

*Remark 2.3.* Note that since we are considering affine connections with respect to a subset of  $\text{Der}(\mathcal{A})$ , it does not make sense in general to demand that  $\nabla_{cd} U = c \nabla_d U$  for (hermitian)  $c \in \mathcal{Z}(\mathcal{A})$ , since  $\mathfrak{g}$  may not be closed under the action of  $\mathcal{Z}(\mathcal{A})$ . However, in the examples we consider it is true that  $\nabla_{cd} U = c \nabla_d U$  whenever  $cd \in \mathfrak{g}$ . (In fact, this is a general statement which follows from Kozul's formula (3.4) as soon as  $\varphi$ , in Definition 3.1, is linear over  $\mathcal{Z}(\mathcal{A})$  in the above sense.)

### 3. PSEUDO-RIEMANNIAN CALCULI

In differential geometry, every derivation of  $C^\infty(M)$  gives rise to a (unique) vector field on the manifold  $M$ . Hence, in the algebraic definition of a connection, where  $\nabla_d U$  is defined for  $d \in \text{Der}(C^\infty(M))$  and  $U \in TM$ , one may swap the two arguments due to the fact that there is a one-to-one correspondence between derivations and vector fields. For instance, this makes the classical definition of torsion meaningful:

$$T(U, V) = \nabla_U V - \nabla_V U - [U, V],$$

from an algebraic point of view. In a derivation based differential calculus over a noncommutative algebra (see e.g. [DV88]), the arguments of a connection is not on

equal footing, partly due to the fact that the set of derivations is in general not a module over the algebra. Thus, there is no natural way to associate an element of the module to an arbitrary derivation.

In this paper, we shall investigate the consequences of introducing a correspondence, which assigns a unique element of a module to every derivation in a Lie algebra  $\mathfrak{g} \subseteq \text{Der}(\mathcal{A})$ . This idea is formalized in the following definition.

**Definition 3.1.** Let  $(M, h)$  be a (right) metric  $\mathcal{A}$ -module, let  $\mathfrak{g} \subseteq \text{Der}(\mathcal{A})$  be a (real) Lie algebra of hermitian derivations and let  $\varphi : \mathfrak{g} \rightarrow M$  be a  $\mathbb{R}$ -linear map. If we denote the pair  $(\mathfrak{g}, \varphi)$  by  $\mathfrak{g}_\varphi$ , the triple  $(M, h, \mathfrak{g}_\varphi)$  is called a *real metric calculus* if

- (1) the image  $M_\varphi = \varphi(\mathfrak{g})$  generates  $M$  as an  $\mathcal{A}$ -module,
- (2)  $h(E, E')^* = h(E, E')$  for all  $E, E' \in M_\varphi$ .

The condition that the elements in the image of  $\varphi$  have hermitian inner products, corresponds to the fact that the metric is real, and that the inner product of two real vector fields, is again a real function. An important consequence of this assumption is that  $h$  is truly symmetric on the image of  $\varphi$ , i.e.  $h(E, E') = h(E', E)$  for all  $E, E' \in M_\varphi$ ; a fact that will repeatedly be used in the sequel.

In this setting, we shall introduce a connection on a real metric calculus, and demand that the connection preserve the hermiticity of  $M_\varphi$ .

**Definition 3.2.** Let  $(M, h, \mathfrak{g}_\varphi)$  be a real metric calculus and let  $\nabla$  denote an affine connection on  $(M, \mathfrak{g})$ . If

$$h(\nabla_d E, E') = h(\nabla_d E, E')^*$$

for all  $E, E' \in M_\varphi$  and  $d \in \mathfrak{g}$  then  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is called a *real connection calculus*.

For a real connection calculus it is straightforward to introduce the concept of a metric and torsion-free connection.

**Definition 3.3.** Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a real connection calculus over  $M$ . The calculus is *metric* if

$$d(h(U, V)) = h(\nabla_d U, V) + h(U, \nabla_d V)$$

for all  $d \in \mathfrak{g}$ ,  $U, V \in M$ , and *torsion-free* if

$$\nabla_{d_1} \varphi(d_2) - \nabla_{d_2} \varphi(d_1) - \varphi([d_1, d_2]) = 0$$

for all  $d_1, d_2 \in \mathfrak{g}$ . A metric and torsion-free real connection calculus over  $M$  is called a *pseudo-Riemannian calculus over  $M$* .

Pseudo-Riemannian calculi will be the main objects of interest to us, and they provide a framework in which one may carry out computations in close analogy with classical Riemannian geometry.

The Levi-Civita theorem states that there is a unique torsion-free and metric connection on the tangent bundle of a Riemannian manifold. In the current setting, one can not guarantee the existence, but given a real metric calculus, there exists at most one connection which is both metric and torsion-free.

**Theorem 3.4.** *Let  $(M, h, \mathfrak{g}_\varphi)$  be a real metric calculus over  $M$ . Then there exists at most one affine connection  $\nabla$  on  $(M, \mathfrak{g})$ , such that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus.*

*Proof.* Assume that there exist two connections  $\nabla$  and  $\tilde{\nabla}$  such that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  and  $(M, h, \mathfrak{g}_\varphi, \tilde{\nabla})$  are pseudo-Riemannian calculi. Let us define

$$\alpha(d, U) = \tilde{\nabla}_d U - \nabla_d U,$$

from which it follows that

$$\alpha(d, Ua) = (\tilde{\nabla}_d U)a + Uda - (\nabla_d U)a - Uda = \alpha(d, U)a,$$

for  $a \in \mathcal{A}$ , as well as

$$\begin{aligned}\alpha(d_1 + \lambda d_2, U) &= \alpha(d_1, U) + \lambda \alpha(d_2, U) \\ \alpha(d, U + V) &= \alpha(d, U) + \alpha(d, V),\end{aligned}$$

for  $\lambda \in \mathbb{R}$ . By subtracting the conditions that  $\nabla, \tilde{\nabla}$  are metric one obtains

$$(3.1) \quad h(\alpha(d, U), V) = -h(U, \alpha(d, V)),$$

and the torsion-free condition implies

$$(3.2) \quad \alpha(d_1, \varphi(d_2)) = \alpha(d_2, \varphi(d_1))$$

for  $d_1, d_2 \in \mathfrak{g}$ . Finally, requiring that  $h(\nabla_{d_1} \varphi(d_2), \varphi(d_3))$  and  $h(\tilde{\nabla}_{d_1} \varphi(d_2), \varphi(d_3))$  are hermitian gives

$$(3.3) \quad h(\alpha(d_1, \varphi(d_2)), \varphi(d_3))^* = h(\alpha(d_1, \varphi(d_2)), \varphi(d_3)).$$

Now, let us make use of (3.1) and (3.2) to compute (where  $E_a = \varphi(d_a)$ )

$$\begin{aligned}h(\alpha(d_1, E_2), E_3) &= h(\alpha(d_2, E_1), E_3) = -h(E_1, \alpha(d_2, E_3)) = -h(E_1, \alpha(d_3, E_2)) \\ &= h(\alpha(d_3, E_1), E_2) = h(\alpha(d_1, E_3), E_2) = -h(E_3, \alpha(d_1, E_2)),\end{aligned}$$

which shows that

$$h(\alpha(d_1, \varphi(d_2)), \varphi(d_3))^* = -h(\alpha(d_1, \varphi(d_2)), \varphi(d_3)).$$

Combining this result with (3.3) yields

$$h(\alpha(d_1, \varphi(d_2)), \varphi(d_3)) = 0,$$

for all  $d_1, d_2, d_3 \in \mathfrak{g}$ . Since the image of  $\varphi$  generates  $M$  and  $h$  is non-degenerate, it follows that  $\alpha(d, U) = 0$  for all  $U \in M$  and  $d \in \mathfrak{g}$ , which shows that

$$\tilde{\nabla}_d U = \nabla_d U$$

for all  $d \in \mathfrak{g}$  and  $U \in M$ . □

The Levi-Civita connection can be explicitly constructed with the help of Kozul's formula, which gives the connection as expressed in terms of the metric tensor. For pseudo-Riemannian calculi, there is a corresponding statement.

**Proposition 3.5.** *Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a pseudo-Riemannian calculus and assume that  $d_1, d_2, d_3 \in \mathfrak{g}$ . Then it holds that*

$$(3.4) \quad \begin{aligned}2h(\nabla_{d_1} E_2, E_3) &= d_1 h(E_2, E_3) + d_2 h(E_1, E_3) - d_3 h(E_1, E_2) \\ &\quad - h(E_1, \varphi([d_2, d_3])) + h(E_2, \varphi([d_3, d_1])) + h(E_3, \varphi([d_1, d_2])),\end{aligned}$$

where  $E_a = \varphi(d_a)$  for  $a \in \{1, 2, 3\}$ .

*Proof.* Using the fact that  $\nabla$  is a metric connection gives

$$(3.5) \quad d_1 h(E_2, E_3) = h(\nabla_{d_1} E_2, E_3) + h(E_2, \nabla_{d_1} E_3)$$

$$(3.6) \quad d_2 h(E_3, E_1) = h(\nabla_{d_2} E_3, E_1) + h(E_3, \nabla_{d_2} E_1)$$

$$(3.7) \quad d_3 h(E_1, E_2) = h(\nabla_{d_3} E_1, E_2) + h(E_1, \nabla_{d_3} E_2),$$

and since the connection is torsion-free one obtains

$$\begin{aligned} h(E_3, \nabla_{d_2} E_1) &= h(E_3, \nabla_{d_1} E_2) + h(E_3, \varphi([d_2, d_1])) \\ h(\nabla_{d_3} E_1, E_2) &= h(\nabla_{d_1} E_3, E_2) + h(\varphi([d_3, d_1]), E_2) \\ h(E_1, \nabla_{d_3} E_2) &= h(E_1, \nabla_{d_2} E_3) + h(E_1, \varphi([d_3, d_2])). \end{aligned}$$

Moreover, using that the connection is real enables us to rewrite the above equations in the following form

$$(3.8) \quad h(E_3, \nabla_{d_2} E_1) = h(\nabla_{d_1} E_2, E_3) + h(E_3, \varphi([d_2, d_1]))$$

$$(3.9) \quad h(\nabla_{d_3} E_1, E_2) = h(E_2, \nabla_{d_1} E_3) + h(\varphi([d_3, d_1]), E_2)$$

$$(3.10) \quad h(E_1, \nabla_{d_3} E_2) = h(\nabla_{d_2} E_3, E_1) + h(E_1, \varphi([d_3, d_2])).$$

Inserting (3.8) in (3.6) and (3.9), (3.10) in (3.7) gives (together with (3.5))

$$\begin{aligned} h(\nabla_{d_1} E_2, E_3) &= d_1 h(E_2, E_3) - h(E_2, \nabla_{d_1} E_3) \\ h(\nabla_{d_1} E_2, E_3) &= d_2 h(E_3, E_1) - h(\nabla_{d_2} E_3, E_1) - h(E_3, \varphi([d_2, d_1])) \\ 0 &= -d_3 h(E_1, E_2) + h(E_2, \nabla_{d_1} E_3) + h(\varphi([d_3, d_1]), E_2) \\ &\quad + h(\nabla_{d_2} E_3, E_1) + h(E_1, \varphi([d_3, d_2])), \end{aligned}$$

and summing these three equations yields

$$\begin{aligned} 2h(\nabla_{d_1} E_2, E_3) &= d_1 h(E_2, E_3) + d_2 h(E_3, E_1) - d_3 h(E_1, E_2) \\ &\quad - h(E_3, \varphi([d_2, d_1])) + h(\varphi([d_3, d_1]), E_2) + h(E_1, \varphi([d_3, d_2])), \end{aligned}$$

which proves (3.4).  $\square$

*Remark 3.6.* Note that Proposition 3.5 gives an independent proof of the fact that the connection is unique, since the hermitian form  $h$  is assumed to be nondegenerate.

Now, let us show that the converse of Proposition (3.4) is true; i.e. a connection satisfying (3.4) gives a pseudo-Riemannian calculus.

**Proposition 3.7.** *Let  $(M, h, \mathfrak{g}_\varphi)$  be a real metric calculus, and let  $\nabla$  be an affine connection on  $(M, \mathfrak{g})$  such that Kozul's formula (3.4) holds. Then  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus.*

*Proof.* From (3.4) it follows immediately that  $h(\nabla_{d_1} \varphi(d_2), \varphi(d_3))$  is hermitian since every term in the right hand side is hermitian, due to the fact that  $(M, h, \mathfrak{g}_\varphi)$  is assumed to be a real metric calculus, which implies that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a real connection calculus. Next, let us show that the connection is metric.

Let  $d_1, d_2, d_3 \in \mathfrak{g}$  and set  $E_a = \varphi(d_a)$ . Using eq. (3.4) twice (together with the fact that  $(M, h, \mathfrak{g}_\varphi)$  is a real metric calculus), gives

$$h(\nabla_{d_1} E_2, E_3) + h(E_2, \nabla_{d_1} E_3) = d_1 h(E_2, E_3).$$

Since  $M_\varphi$  generates  $M$ , one may find  $\{E_a = \varphi(d_a)\}_{a=1}^N$  such that one can write  $U = E_a U^a$  for all  $U \in M$ . It then follows that

$$\begin{aligned} & h(\nabla_d U, V) + h(U, \nabla_d V) \\ &= h((\nabla_d E_a)U^a + E_a dU^a, E_b V^b) + h(E_a U^a, (\nabla_d E_b)V^b + E_b dV^b) \\ &= (U^a)^* (h(\nabla_d E_a, E_b) + h(E_a, \nabla_d E_b)) V^b + d(U^a)^* h(E_a, E_b) V^b + (U^a)^* h(E_a, E_b) dV^b \\ &= (U^a)^* dh(E_a, E_b) V^b + d(U^a)^* h(E_a, E_b) V^b + (U^a)^* h(E_a, E_b) dV^b \\ &= d((U^a)^* h(E_a, E_b) V^b) = dh(U, V), \end{aligned}$$

which shows that the affine connection is metric. Finally, let us show that the connection is torsion-free. For  $d_1, d_2, d_3 \in g$ , with  $E_a = \varphi(d_a)$ , let us consider

$$T = h(\nabla_{d_1} E_2 - \nabla_{d_2} E_1 - \varphi([d_1, d_2]), E_3).$$

By using formula (3.4) for the first two terms, one obtains

$$T = h(E_3, \varphi([d_1, d_2])) - h(\varphi([d_1, d_2]), E_3) = 0.$$

Since the image of  $\varphi$  generates  $M$  one can conclude that

$$h(\nabla_{d_1} E_2 - \nabla_{d_2} E_1 - \varphi([d_1, d_2]), U) = 0$$

for all  $U \in M$ , which implies that

$$\nabla_{d_1} E_2 - \nabla_{d_2} E_1 - \varphi([d_1, d_2]) = 0,$$

since  $h$  is nondegenerate.  $\square$

In particular examples, it is possible to use Kozul's formula to construct a metric and torsion-free connection. One of the cases, which is relevant to our examples, is when  $M$  is a free module.

**Corollary 3.8.** *Let  $(M, h, \mathfrak{g}_\varphi)$  be a real metric calculus and let  $\{\partial_1, \dots, \partial_n\}$  be a basis of  $\mathfrak{g}$  such that  $\{E_a = \varphi(\partial_a)\}_{a=1}^n$  is a basis for  $M$ . If there exist  $U_{ab} \in M$  such that*

$$(3.11) \quad \begin{aligned} 2h(U_{ab}, E_c) &= \partial_a h(E_b, E_c) + \partial_b h(E_a, E_c) - \partial_c h(E_a, E_b) \\ &\quad - h(E_a, \varphi([\partial_b, \partial_c])) + h(E_b, \varphi([\partial_c, \partial_a])) + h(E_c, \varphi([\partial_a, \partial_b])) \end{aligned}$$

for  $a, b, c = 1, \dots, n$ , then there exists a connection  $\nabla$ , given by  $\nabla_{\partial_a} E_b = U_{ab}$ , such that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus.

*Proof.* Assuming that such elements  $U_{ab} \in M$  exist, define

$$\nabla_{\partial_a} E_b = U_{ab}$$

and extend  $\nabla$  to  $\mathfrak{g}$  by linearity. Since  $\{E_a\}_{a=1}^n$  is a basis of  $M$ , every element  $U \in M$  has a unique expression  $U = E_a U^a$ , and we extend  $\nabla$  to  $M$  through linearity and Leibniz' rule

$$\nabla_d U = (\nabla_d E_a) U^a + E_a d(U^a),$$

which then defines an affine connection on  $(M, \mathfrak{g})$ . From Proposition 3.7 it follows that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus.  $\square$

## 4. CURVATURE OF PSEUDO-RIEMANNIAN CALCULI

In this section we will study symmetries of the curvature tensor of a pseudo-Riemannian calculus, as well as introduce an associated scalar curvature, in case it exists. It turns out that in order to recover the full symmetry (compared to the classical setting) of the curvature tensor, one needs an extra assumption of hermiticity. Namely, although a real connection calculus satisfies the requirement that  $h(\nabla_{d_1} E_1, E_2)$  is hermitian, there is no guarantee that  $h(\nabla_{d_1} \nabla_{d_2} E_1, E_2)$  is hermitian. However, with this extra assumption, one may prove that all the familiar symmetries of the curvature tensor hold (cf. Proposition 4.5). Pseudo-Riemannian calculi fulfilling this extra condition will appear often in what follows, and therefore we make the following definition.

**Definition 4.1.** A pseudo-Riemannian calculus  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is said to be *real* if  $h(\nabla_{d_1} \nabla_{d_2} E_1, E_2)$  is hermitian for all  $d_1, d_2 \in \mathfrak{g}$  and  $E_1, E_2 \in M_\varphi$ .

For later convenience, let us provide a slight reformulation of the condition in the definition above.

**Lemma 4.2.** *Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a pseudo-Riemannian calculus. Then the following statements are equivalent*

- (1)  $h(\nabla_{d_1} \nabla_{d_2} E_1, E_2)$  is hermitian for all  $d_1, d_2 \in \mathfrak{g}$  and  $E_1, E_2 \in M_\varphi$ ,
- (2)  $h(\nabla_{d_1} E_1, \nabla_{d_2} E_2)$  is hermitian for all  $d_1, d_2 \in \mathfrak{g}$  and  $E_1, E_2 \in M_\varphi$ .

*Proof.* Since the connection is metric, one may write

$$d_2 h(\nabla_{d_1} E_1, E_2) = h(\nabla_{d_2} \nabla_{d_1} E_1, E_2) + h(\nabla_{d_1} E_1, \nabla_{d_2} E_2)$$

Now,  $d_2 h(\nabla_{d_1} E_1, E_2)$  is hermitian (since  $\nabla$  is real and  $d_2$  is hermitian), and it follows that if one of  $h(\nabla_{d_2} \nabla_{d_1} E_1, E_2)$  and  $h(\nabla_{d_1} E_1, \nabla_{d_2} E_2)$  is hermitian, then the other one is hermitian too (since it is then a sum of two hermitian elements).  $\square$

In a pseudo-Riemannian calculus  $(M, h, \mathfrak{g}_\varphi, \nabla)$ , one introduces the curvature operator in a standard way as

$$R(d_1, d_2)U = \nabla_{d_1} \nabla_{d_2} U - \nabla_{d_2} \nabla_{d_1} U - \nabla_{[d_1, d_2]} U$$

for  $d_1, d_2 \in \mathfrak{g}$  and  $U \in M$ . The operator  $R(d_1, d_2)$  has a trivial antisymmetry when exchanging its arguments  $d_1, d_2$  and, furthermore, due to the torsion-free condition, the first Bianchi identity holds.

**Proposition 4.3.** *Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a pseudo-Riemannian calculus with curvature operator  $R$ . Then*

- (1)  $h(U, R(d_1, d_2)V) = -h(U, R(d_2, d_1)V)$
- (2)  $R(d_1, d_2)\varphi(d_3) + R(d_2, d_3)\varphi(d_1) + R(d_3, d_1)\varphi(d_2) = 0,$

for  $U, V \in M$  and  $d_1, d_2, d_3 \in \mathfrak{g}$ .



*Proof.* Property (1) follows immediately from the definition of the curvature operator. To prove (2), one uses the torsion free condition twice (set  $E_a = \varphi(d_a)$ ):

$$\begin{aligned}
& R(d_1, d_2)E_3 + R(d_2, d_3)E_1 + R(d_3, d_1)E_2 \\
&= \nabla_{d_1}(\nabla_{d_2}E_3 - \nabla_{d_3}E_2) + \nabla_{d_2}(\nabla_{d_3}E_1 - \nabla_{d_1}E_3) + \nabla_{d_3}(\nabla_{d_1}E_2 - \nabla_{d_2}E_1) \\
&\quad - \nabla_{[d_1, d_2]}E_3 - \nabla_{[d_2, d_3]}E_1 - \nabla_{[d_3, d_1]}E_2 \\
&= \nabla_{d_1}\varphi([d_2, d_3]) + \nabla_{d_2}\varphi([d_3, d_1]) + \nabla_{d_3}\varphi([d_1, d_2]) \\
&\quad - \nabla_{[d_1, d_2]}E_3 - \nabla_{[d_2, d_3]}E_1 - \nabla_{[d_3, d_1]}E_2 \\
&= \varphi([d_1, [d_2, d_3]]) + \varphi([d_2, [d_3, d_1]]) + \varphi([d_3, [d_1, d_2]]) = 0,
\end{aligned}$$

where the last equality follows from the Jacobi identity, and the fact that  $\varphi$  is a linear map.  $\square$

As already mentioned, the full symmetry of the curvature operator is recovered in the case of *real* pseudo-Riemannian calculi. This is stated in Proposition 4.5, and in the proof we shall need the following short lemma.

**Lemma 4.4.** *If  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus, then*

$$d(h(E, E)) = 2h(E, \nabla_d E)$$

for all  $d \in \mathfrak{g}$  and  $E \in M_\varphi$ .

*Proof.* Since  $\nabla$  is a metric connection

$$d(h(E, E)) = h(\nabla_d E, E) + h(E, \nabla_d E),$$

and, as  $\nabla$  is real, it follows that  $h(E, \nabla_d E) = h(\nabla_d E, E)$ , which implies that

$$d(h(E, E)) = 2h(E, \nabla_d E)$$

for all  $E \in M_\varphi$  and  $d \in \mathfrak{g}$ .  $\square$

Note that, for the sake of completeness, the results of Proposition 4.3 are repeated in the formulation below.

**Proposition 4.5.** *Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a real pseudo-Riemannian calculus, with curvature operator  $R$ . Then*

- (a)  $h(U, R(d_1, d_2)V) = -h(U, R(d_2, d_1)V)$ ,
- (b)  $h(E_1, R(d_1, d_2)E_2) = -h(E_2, R(d_1, d_2)E_1)$ ,
- (c)  $R(d_1, d_2)\varphi(d_3) + R(d_2, d_3)\varphi(d_1) + R(d_3, d_1)\varphi(d_2) = 0$ ,
- (d)  $h(\varphi(d_1), R(d_3, d_4)\varphi(d_2)) = h(\varphi(d_3), R(d_1, d_2)\varphi(d_4))$ ,

for all  $U, V \in M$ ,  $E_1, E_2 \in M_\varphi$  and  $d_1, d_2, d_3, d_4 \in \mathfrak{g}$ .

*Proof.* Properties (a) and (c) are contained in the statement of Proposition 4.3, which is valid for an arbitrary pseudo-Riemannian calculus. Let now show that (b) holds, by proving that  $h(E, R(d_1, d_2)E) = 0$  for all  $E \in M_\varphi$ . By using the fact that  $\nabla$  is metric, one computes

$$\begin{aligned}
h(E, R(d_1, d_2)E) &= h(E, \nabla_{d_1}\nabla_{d_2}E - \nabla_{d_2}\nabla_{d_1}E - \nabla_{[d_1, d_2]}E) \\
&= d_1h(E, \nabla_{d_2}E) - d_2h(E, \nabla_{d_1}E) - h(E, \nabla_{[d_1, d_2]}E),
\end{aligned}$$

using the result in Lemma 4.2 (and the fact that the pseudo-Riemannian calculus is assumed to be real). Next, it follows from Lemma 4.4 that

$$h(E, R(d_1, d_2)E) = \frac{1}{2}d_1d_2h(E, E) - \frac{1}{2}d_2d_1h(E, E) - \frac{1}{2}[d_1, d_2]h(E, E) = 0.$$

Finally, we prove (d) by using (c) to write (again,  $E_a = \varphi(d_a)$ )

$$\begin{aligned}
0 &= h(E_1, R(d_2, d_3)E_4 + R(d_3, d_4)E_2 + R(d_4, d_2)E_3) \\
&\quad + h(E_2, R(d_3, d_4)E_1 + R(d_4, d_1)E_3 + R(d_1, d_3)E_4) \\
&\quad + h(E_3, R(d_4, d_1)E_2 + R(d_1, d_2)E_4 + R(d_2, d_4)E_1) \\
&\quad + h(E_4, R(d_1, d_2)E_3 + R(d_2, d_3)E_1 + R(d_3, d_1)E_2) \\
&= 2h(E_1, R(d_4, d_2)E_3) + 2h(E_2, R(d_1, d_3)E_4),
\end{aligned}$$

by using (b) and (a). Consequently, by using (a) once more, relation (d) follows.  $\square$

**4.1. Scalar curvature.** Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a real pseudo-Riemannian calculus, and let  $\{\partial_1, \dots, \partial_n\}$  be a basis of  $\mathfrak{g}$ . Setting  $E_a = \varphi(\partial_a)$  one introduces the components of the metric and the curvature tensor relative to this basis via

$$\begin{aligned}
h_{ab} &= h(E_a, E_b) \\
R_{abpq} &= h(E_a, R(\partial_p, \partial_q)E_b),
\end{aligned}$$

and we note that  $(R_{abpq})^* = R_{abpq}$  (using the fact that the pseudo-Riemannian calculus is real). Proposition 4.5 implies that

$$(4.1) \quad R_{abpq} = -R_{abqp},$$

$$(4.2) \quad R_{abpq} = -R_{bapq},$$

$$(4.3) \quad R_{abpq} = R_{pqab},$$

$$(4.4) \quad R_{apqr} + R_{aqrp} + R_{arqp} = 0.$$

In the traditional definition of scalar curvature  $S = h^{ab}h^{pq}R_{apbq}$  one makes use of the inverse of the metric to contract indices of the curvature tensor. For an arbitrary algebra, the metric  $h_{ab}$  may fail to be invertible; i.e., there does not exist  $h^{ab}$  such that  $h^{ab}h_{bc} = \delta_c^a \mathbf{1}$ . However, one might be in the situation where there exist  $H \in \mathcal{A}$  and  $\hat{h}^{ab}$  such that

$$\hat{h}^{ab}h_{bc} = h_{cb}\hat{h}^{ba} = \delta_c^a H.$$

If  $H$  is hermitian and regular (i.e. not a zero divisor), then we say that  $h_{ab}$  has a pseudo-inverse  $(\hat{h}^{ab}, H)$ .

**Lemma 4.6.** *If  $(\hat{h}^{ab}, H)$  and  $(\hat{g}^{ab}, G)$  are pseudo-inverses for  $h_{ab}$  then*

- (1) *if  $G = H$  then  $\hat{g}^{ab} = \hat{h}^{ab}$ ,*
- (2)  *$[h_{ab}, H] = [\hat{h}^{ab}, H] = 0$ ,*
- (3)  *$(\hat{h}^{ab})^* = \hat{h}^{ba}$ ,*
- (4)  *$\hat{g}^{ab}H = G\hat{h}^{ab}$  and  $H\hat{g}^{ab} = \hat{h}^{ab}G$ ,*
- (5) *if  $[H, \hat{h}^{ab}] = 0$  then  $[G, \hat{h}^{ab}] = [H, G] = 0$ .*

*Proof.* To prove (1), one assumes that  $(\hat{h}, H)$  and  $(\hat{g}, H)$  are pseudo-inverses of  $h$ . Then it follows that  $(\hat{g}^{ab} - \hat{h}^{ab})h_{bc} = 0$  which, when multiplying from the right by  $\hat{h}^{cp}$ , yields

$$(\hat{g}^{ap} - \hat{h}^{ap})H = 0.$$

Since  $H$  is a regular element it follows that  $\hat{g}^{ap} = \hat{h}^{ap}$ .

By using the definition of the two pseudo-inverses, one may rewrite the expression  $\hat{g}^{ab}h_{bc}\hat{h}^{cp}$  in two ways,

$$\begin{aligned} (\hat{g}^{ab}h_{bc})\hat{h}^{cp} &= G\delta_c^a\hat{h}^{cp} = G\hat{h}^{ap} \\ \hat{g}^{ab}(h_{bc}\hat{h}^{cp}) &= \hat{g}^{ab}H\delta_b^p = \hat{g}^{ap}H \end{aligned}$$

proving (4) (consider  $\hat{h}^{ab}h_{bc}\hat{g}^{cp}$  for the second part of the statement). Setting  $(\hat{h}, H) = (\hat{g}, G)$  in the above result immediately gives  $[H, \hat{h}^{ab}] = 0$ . Together with

$$h_{ab}H = h_{ap}\delta_b^p H = h_{ap}\hat{h}^{pc}h_{cb} = H\delta_a^c h_{cb} = Hh_{ab}$$

this proves (2).

Let us consider property (5). If  $[H, \hat{g}^{ab}] = 0$  then (4) implies that

$$G\hat{h}^{ab} = \hat{g}^{ab}H = H\hat{g}^{ab} = \hat{h}^{ab}G.$$

Moreover,

$$\begin{aligned} \hat{g}^{ab}H - H\hat{g}^{ab} &= 0 \Rightarrow h_{ca}\hat{g}^{ab}H - h_{ca}H\hat{g}^{ab} = 0 \Rightarrow \text{(using (2))} \\ h_{ca}\hat{g}^{ab}H - Hh_{ca}\hat{g}^{ab} &= 0 \Rightarrow (GH - HG)\delta_c^b = 0 \Rightarrow [G, H] = 0, \end{aligned}$$

which concludes the proof of (5).

Finally, to prove (3) one considers the hermitian conjugates of  $h_{ab}\hat{h}^{bc} = H\delta_a^c$  and  $\hat{h}^{ab}h_{bc} = H\delta_c^a$ , which gives

$$\begin{aligned} (\hat{h}^{bc})^* h_{ba} &= H\delta_a^c \\ h_{cb}(\hat{h}^{ab})^* &= H\delta_c^a, \end{aligned}$$

by using that  $h_{ab}^* = h_{ba}$ . The above equations show that if  $k^{ab} = (\hat{h}^{ba})^*$  then  $(k^{ab}, H)$  is a pseudo-inverse for  $h_{ab}$ . Since  $(\hat{h}^{ab}, H)$  and  $(k^{ab}, H)$  are pseudo-inverses for  $h_{ab}$ , it follows from (1) that  $\hat{h}^{ab} = k^{ab} = (\hat{h}^{ba})^*$ .  $\square$

**Definition 4.7.** Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a real pseudo-Riemannian calculus such that  $h_{ab}$  has a pseudo-inverse  $(\hat{h}^{ab}, H)$  with respect to a basis of  $\mathfrak{g}$ . A *scalar curvature of  $(M, h, \mathfrak{g}_\varphi, \nabla)$  with respect to  $(\hat{h}^{ab}, H)$*  is an element  $S \in \mathcal{A}$  such that

$$\hat{h}^{ab}R_{apbq}\hat{h}^{pq} = HSH.$$

*Remark 4.8.* Note that it is easy to show that  $\hat{h}^{ab}R_{apbq}\hat{h}^{pq}$  and, hence, the scalar curvature with respect to  $(\hat{h}^{ab}, H)$ , is independent of the choice of basis in  $\mathfrak{g}$ .

**Proposition 4.9.** Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a real pseudo-Riemannian calculus, and let  $(\hat{h}^{ab}, H)$  be a pseudo-inverse of  $h_{ab}$  with respect to a basis of  $\mathfrak{g}$ . Then there exists at most one scalar curvature of  $(M, h, \mathfrak{g}_\varphi, \nabla)$  with respect to  $(\hat{h}^{ab}, H)$  and, furthermore, the scalar curvature is hermitian.

*Proof.* Uniqueness of the scalar curvature follows immediately from the fact that  $H$  is regular; namely,

$$HSH = HS'H \Leftrightarrow H(S - S')H = 0,$$

which then implies that  $S = S'$  by the regularity of  $H$ .

As noted in the beginning of this section,  $R_{abcd}$  is hermitian. Furthermore, Lemma 4.6 states that  $(\hat{h}^{ab})^* = \hat{h}^{ba}$  which implies that

$$(\hat{h}^{ab}R_{apbq}\hat{h}^{pq})^* = \hat{h}^{qp}R_{apbq}\hat{h}^{ba} = \hat{h}^{qp}R_{qbpa}\hat{h}^{ba} = \hat{h}^{ab}R_{apbq}\hat{h}^{pq},$$

by using (4.1)–(4.3). From the definition of scalar curvature, this implies that

$$HSH = (HSH)^* = HS^*H \Leftrightarrow H(S - S^*)H = 0.$$

Since  $H$  is assumed to be regular, it follows that  $S = S^*$ .  $\square$

If  $S$  is the scalar curvature with respect to a pseudo-inverse  $(\hat{h}^{ab}, H)$ , in which  $H$  is central, then any scalar curvature (with respect to an arbitrary pseudo-inverse) coincides with  $S$ , giving a unique hermitian scalar curvature of a real pseudo-Riemannian calculus.

**Proposition 4.10.** *Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be a real pseudo-Riemannian calculus with scalar curvature  $S$  with respect to  $(\hat{h}^{ab}, H)$ . If  $H \in \mathcal{Z}(\mathcal{A})$  then the scalar curvature is unique; i.e. if  $S'$  is the scalar curvature with respect to  $(\hat{g}^{ab}, G)$ , then  $S' = S$ .*

*Proof.* If  $H \in \mathcal{Z}(\mathcal{A})$  then property (4) of Lemma 4.6 implies that

$$G(HSH)G = G\hat{h}^{ab}R_{apbq}\hat{h}^{pq}G = H\hat{g}^{ab}R_{apbq}\hat{g}^{pq}H = H(GS'G)H,$$

and since  $[H, G] = 0$  one obtains

$$HG(S - S')GH = 0 \Rightarrow S = S'$$

since  $G$  and  $H$  are assumed to be regular.  $\square$

*Remark 4.11.* In particular, if the metric  $h_{ab}$  is invertible, i.e. it has a pseudo-inverse  $(\hat{h}^{ab}, \mathbb{1})$ , then Proposition 4.10 implies that there exists a unique scalar curvature of the corresponding real pseudo-Riemannian calculus.

## 5. THE NONCOMMUTATIVE TORUS

After having developed a general framework for Riemannian curvature of a real metric calculus, it is time to consider some examples, in order to motivate our definitions. As a starter, let us consider the noncommutative torus and construct a pseudo-Riemannian calculus over it (cf. [Arn14] for a related approach that uses the concrete embedding of the torus into  $\mathbb{R}^4$ ). For the noncommutative torus, our construction of a Levi-Civita connection, and its corresponding curvature, is similar to the approach taken in [Ros13].

As we shall work in close analogy with differential geometry, let us briefly review the geometry of the Clifford torus. We consider the Clifford torus as embedded in  $\mathbb{R}^4$  with the flat induced metric. Concretely, let us consider the following parametrization

$$\vec{x} = (x^1, x^2, x^3, x^4) = (\cos u, \sin u, \cos v, \sin v)$$

which implies that the tangent space at a point is spanned by

$$\partial_u \vec{x} = (-\sin u, \cos u, 0, 0) = (-x^2, x^1, 0, 0)$$

$$\partial_v \vec{x} = (0, 0, -\sin v, \cos v) = (0, 0, -x^4, x^3),$$

from which the induced metric is obtained as

$$(h_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Setting  $z = x^1 + ix^2$ ,  $w = x^3 + ix^4$  and  $\partial_1 = \partial_u$ ,  $\partial_2 = \partial_v$  yields

$$\begin{aligned} \partial_1 z &= iz & \partial_1 w &= 0 \\ \partial_2 z &= 0 & \partial_2 w &= iw. \end{aligned}$$

As the noncommutative torus  $T_\theta^2$ , we consider the unital  $*$ -algebra generated by two unitary operators  $Z, W$  satisfying  $WZ = qZW$  with  $q = e^{2\pi i\theta}$ , and we introduce

$$\begin{aligned} X^1 &= \frac{1}{2}(Z + Z^*) & X^2 &= \frac{1}{2i}(Z - Z^*) \\ X^3 &= \frac{1}{2}(W + W^*) & X^4 &= \frac{1}{2i}(W - W^*). \end{aligned}$$

In analogy with the geometrical setting, let  $M$  be the (right) submodule of  $(T_\theta^2)^4$  generated by

$$\begin{aligned} E_1 &= (-X^2, X^1, 0, 0) \\ E_2 &= (0, 0, -X^4, X^3), \end{aligned}$$

and for  $U, V \in M$ , with  $U = E_a U^a$  and  $V = E_a V^a$  we set

$$h(U, V) = \sum_{a=1}^2 (U^a)^* V^a.$$

**Proposition 5.1.** *The elements  $E_1, E_2 \in M$  gives a basis for  $M$  and  $h$  is a non-degenerate hermitian form on  $M$ . Thus,  $(M, h)$  is a free metric  $T_\theta^2$ -module.*

*Proof.* First, let us show that  $E_1, E_2$  are free generators

$$\begin{aligned} E_1 a + E_2 b = 0 &\Rightarrow (-X^2 a, X^1 a, -X^4 b, X^3 b) = (0, 0, 0, 0) \Rightarrow \\ \begin{cases} ((X^1)^2 + (X^2)^2)a = 0 \\ ((X^3)^2 + (X^4)^2)b = 0 \end{cases} &\Leftrightarrow \begin{cases} ZZ^* a = 0 \\ WW^* b = 0 \end{cases} \Leftrightarrow a = b = 0. \end{aligned}$$

Next, we prove that  $h$  is nondegenerate on  $M$ . Let  $U, V \in M$  and write  $U = E_a U^a$ . Assuming that  $h(U, V) = 0$  for all  $V \in M$  may be equivalently stated as  $h(U, E_a) = 0$  for  $a = 1, 2$ , which immediately gives  $U^1 = U^2 = 0$ .  $\square$

Next, we let  $\mathfrak{g}$  be the (real) Lie algebra generated by the two hermitian derivations  $\partial_1, \partial_2$ , given by

$$\begin{aligned} \partial_1 Z &= iZ & \partial_1 W &= 0 \\ \partial_2 Z &= 0 & \partial_2 W &= iW, \end{aligned}$$

from which it follows that  $[\partial_1, \partial_2] = 0$ . Together with the map  $\varphi : \mathfrak{g} \rightarrow M$ , defined as  $\varphi(\partial_a) = E_a$  and extended by linearity, it is easy to check that  $(M, h, \mathfrak{g}_\varphi)$  is a real metric calculus over  $T_\theta^2$ . Furthermore, we note that, with respect to the basis  $\{\partial_1, \partial_2\}$  of  $\mathfrak{g}$ , the metric

$$(h_{ab}) = (h(E_a, E_b)) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

is invertible.

One is now in position to use Corollary 3.8 to find a unique connection  $\nabla$  on  $M$  such that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus. However, since  $h(E_a, E_b) = \delta_{ab}\mathbf{1}$  and  $[\partial_a, \partial_b] = 0$ , the only solution to

$$\begin{aligned} 2h(U_{ab}, E_c) &= \partial_a h(E_b, E_c) + \partial_b h(E_a, E_c) - \partial_c h(E_a, E_b) \\ &\quad - h(E_a, \varphi([\partial_b, \partial_c])) + h(E_b, \varphi([\partial_c, \partial_a])) + h(E_c, \varphi([\partial_a, \partial_b])) \end{aligned}$$

is  $U_{ab} = 0$ , which gives  $\nabla_d U = 0$  for all  $d \in \mathfrak{g}$  and  $U \in M$ . Hence, the curvature of the corresponding pseudo-Riemannian calculus vanishes identically, and the (unique) scalar curvature is 0.

As done in [Ros13] one can obtain more interesting results by conformally perturbing the metric

$$h_\alpha(U, V) = \sum_{a=1}^2 (U^a)^* e^\alpha V^a$$

for some hermitian element  $\alpha \in T_\theta^2$  (here, of course, one considers the smooth part of the  $C^*$ -algebra generated by  $Z, W$ ). One can easily check that  $(M, h_\alpha, \mathfrak{g}_\varphi)$  is a real metric calculus, and one may find a connection  $\nabla$  (using Corollary 3.8) such that  $(M, h_\alpha, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus. However, unless  $\alpha$  is central, it will in general not be a *real* pseudo-Riemannian calculus.

## 6. THE NONCOMMUTATIVE 3-SPHERE

As a main motivating example for this paper, we consider the noncommutative 3-sphere. We shall explicitly construct a real pseudo-Riemannian calculus together with a basis of  $\mathfrak{g}$  for which the metric  $h_{ab}$  has a pseudo-inverse  $(\hat{h}^{ab}, H)$  with  $H \in \mathcal{Z}(\mathcal{A})$ , giving a unique scalar curvature (via Proposition 4.10). As for the case of the torus, we shall use the analogy with differential geometry to find an appropriate metric calculus. Therefore, let us start by recalling the Hopf parametrization of  $S^3$ .

**6.1. Hopf coordinates for  $S^3$ .** The 3-sphere can be described as embedded in  $\mathbb{C}^2$  by two complex coordinates  $z = x^1 + ix^2$  and  $w = x^3 + ix^4$ , satisfying  $|z|^2 + |w|^2 = 1$ , which can be realized by

$$\begin{aligned} z &= e^{i\xi_1} \sin \eta \\ w &= e^{i\xi_2} \cos \eta, \end{aligned}$$

giving

$$\begin{aligned} x^1 &= \cos \xi_1 \sin \eta & x^2 &= \sin \xi_1 \sin \eta \\ x^3 &= \cos \xi_2 \cos \eta & x^4 &= \sin \xi_2 \cos \eta. \end{aligned}$$

At every point where  $0 < \xi_1, \xi_2 < 2\pi$  and  $0 < \eta < \pi/2$ , the tangent space is spanned by the three vectors

$$\begin{aligned} E_1 &= \partial_1(x^1, x^2, x^3, x^4) = (-x^2, x^1, 0, 0) \\ E_2 &= \partial_2(x^1, x^2, x^3, x^4) = (0, 0, -x^4, x^3) \\ E_\eta &= \partial_\eta(x^1, x^2, x^3, x^4) = (\cos \xi_1 \cos \eta, \sin \xi_1 \cos \eta, -\cos \xi_2 \sin \eta, -\sin \xi_2 \sin \eta), \end{aligned}$$

where  $\partial_1 = \partial_{\xi_1}$  and  $\partial_2 = \partial_{\xi_2}$ . Instead of  $\partial_\eta$ , one may introduce the derivation  $\partial_3 = |z||w|\partial_\eta$ , which gives

$$E_3 = \partial_3(x^1, x^2, x^3, x^4) = (x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2),$$

and one may equally well span the tangent space by  $E_1, E_2, E_3$ . The action of  $\partial_1, \partial_2, \partial_3$  on  $z$  and  $w$  is given by

$$(6.1) \quad \partial_1(z) = iz \quad \partial_1(w) = 0$$

$$(6.2) \quad \partial_2(z) = 0 \quad \partial_2(w) = iw$$

$$(6.3) \quad \partial_3(z) = z|w|^2 \quad \partial_3(w) = -w|z|^2.$$

With respect to the basis  $\{E_1, E_2, E_3\}$  of  $T_p S^3$  the induced metric becomes

$$(6.4) \quad (h_{ab}) = \begin{pmatrix} |z|^2 & 0 & 0 \\ 0 & |w|^2 & 0 \\ 0 & 0 & |z|^2|w|^2 \end{pmatrix}.$$

**6.2. A pseudo-Riemannian calculus for  $S_\theta^3$ .** For our purposes, the noncommutative three sphere  $S_\theta^3$  [Mat91a, Mat91b] is a unital  $*$ -algebra generated by  $Z, Z^*, W, W^*$  subject to the relations

$$(6.5) \quad \begin{aligned} WZ &= qZW & W^*Z &= \bar{q}ZW^* & WZ^* &= \bar{q}Z^*W & W^*Z^* &= qZ^*W^* \\ ZZ^* &= ZZ^* & WW^* &= WW^* & WW^* &= \mathbf{1} - ZZ^*, \end{aligned}$$

where  $q = e^{2\pi i\theta}$ . It follows from the defining relations that a basis for  $S_\theta^3$  is given by the monomials

$$Z^i (Z^*)^j W^{(k)}$$

for  $i, j \geq 0$  and  $k \in \mathbb{Z}$ , where

$$W^{(k)} = \begin{cases} W^k & \text{if } k \geq 0 \\ (W^*)^{-k} & \text{if } k < 0 \end{cases}.$$

Let us collect a few properties of  $S_\theta^3$  that will be useful to us.

**Proposition 6.1.** *If  $a \in S_\theta^3$  then*

$$\begin{aligned} (1) \quad ZZ^*a = 0 &\Rightarrow a = 0, \\ (2) \quad WW^*a = 0 &\Rightarrow a = 0. \end{aligned}$$

*Moreover,  $ZZ^*$  and  $WW^*$  are central elements of  $S_\theta^3$ .*

*Proof.* Let us prove that  $ZZ^*$  commutes with every element of  $S_\theta^3$  (the proof for  $WW^*$  is analogous). From the defining relations of the algebra, it is clear that  $ZZ^*$  commutes with  $Z$  and  $Z^*$ . Let us check that  $ZZ^*$  commutes with  $W$  and  $W^*$ :

$$\begin{aligned} WZZ^* &= qZWZ^* = q\bar{q}ZZ^*W = ZZ^*W \\ W^*ZZ^* &= \bar{q}ZW^*Z^* = \bar{q}qZZ^*W^* = ZZ^*W^*. \end{aligned}$$

Next, let us show that neither  $ZZ^*$  nor  $WW^*$  is a zero divisor. An arbitrary element  $a \in S_\theta^3$  may be written as

$$a = \sum_{i,j \geq 0, k \in \mathbb{Z}} a_{ijk} Z^i (Z^*)^j W^{(k)}$$

for  $a_{ijk} \in \mathbb{C}$ , and it follows that

$$ZZ^*a = \sum_{i,j \geq 0, k \in \mathbb{Z}} a_{ijk} Z^{i+1} (Z^*)^{j+1} W^{(k)}$$

since  $[Z, Z^*] = 0$ . As  $Z^i(Z^*)^j W^{(k)}$  is a basis for  $S_\theta^3$ , setting  $ZZ^*a = 0$  demands that  $a_{ijk} = 0$  for all  $i, j \geq 0$  and  $k \in \mathbb{Z}$ , which implies that  $a = 0$ . Analogously,

$$\begin{aligned} WW^*a &= (\mathbf{1} - ZZ^*)a = \sum_{i,j \geq 0, k \in \mathbb{Z}} \left( a_{ijk} Z^i(Z^*)^j W^{(k)} - a_{ijk} Z^{i+1}(Z^*)^{j+1} W^{(k)} \right) \\ &= \sum_{j \geq 0, k \in \mathbb{Z}} a_{0jk} (Z^*)^j W^{(k)} + \sum_{i \geq 1, k \in \mathbb{Z}} a_{i0k} Z^i W^{(k)} \\ &\quad + \sum_{i,j \geq 1, k \in \mathbb{Z}} (a_{ijk} - a_{i-1,j-1,k}) Z^i(Z^*)^j W^{(k)}, \end{aligned}$$

which can easily be seen to give  $a_{ijk} = 0$  upon setting  $WW^*a = 0$ .  $\square$

Let us introduce the notation

$$\begin{aligned} X^1 &= \frac{1}{2}(Z + Z^*) & X^2 &= \frac{1}{2i}(Z - Z^*) \\ X^3 &= \frac{1}{2}(W + W^*) & X^4 &= \frac{1}{2i}(W - W^*) \\ |Z|^2 &= ZZ^* & |W|^2 &= WW^*, \end{aligned}$$

and note that  $|Z|^2 = (X^1)^2 + (X^2)^2$  and  $|W|^2 = (X^3)^2 + (X^4)^2$ , as well as

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = |Z|^2 + |W|^2 = \mathbf{1}.$$

In the following, we shall construct a real pseudo-Riemannian calculus for  $S_\theta^3$ . Let us start by introducing a metric module  $(M, h)$  in close analogy with the Hopf parametrization in Section 6.1. Therefore, we let  $E_1, E_2, E_3$  be the following elements of the free (right) module  $(S_\theta^3)^4$ :

$$\begin{aligned} E_1 &= (-X^2, X^1, 0, 0) \\ E_2 &= (0, 0, -X^4, X^3) \\ E_3 &= (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2), \end{aligned} \tag{6.6}$$

and let  $M$  be the module generated by  $\{E_1, E_2, E_3\}$ .

**Proposition 6.2.** *The module  $M = \{E_1a + E_2b + E_3c : a, b, c \in S_\theta^3\}$  is a free right  $S_\theta^3$ -module with a basis given by the elements  $\{E_1, E_2, E_3\}$ .*

*Proof.* By construction  $\{E_1, E_2, E_3\}$  are generators of  $M$ . To prove that  $M$  is a free module, we assume that  $a, b, c \in S_\theta^3$  is such that

$$E = E_1a + E_2b + E_3c = 0,$$

and show that  $a = b = c = 0$ . The requirement that  $E = 0$  is equivalent to

$$\begin{aligned} -X^2a + X^1|W|^2c &= 0 & X^1a + X^2|W|^2c &= 0 \\ -X^4b - X^3|Z|^2c &= 0 & X^3b - X^4|Z|^2c &= 0, \end{aligned}$$

and multiplying the first two equations by  $X^1$  and  $X^2$ , respectively, and summing them yields (using that  $[X^1, X^2] = 0$ )

$$((X^1)^2 + (X^2)^2)|W|^2c = 0 \quad \Leftrightarrow \quad |Z|^2|W|^2c = 0.$$

It follows from Proposition 6.1 that  $c = 0$ , and the system of equations becomes

$$\begin{aligned} X^2a &= 0 & X^1a &= 0 \\ X^4b &= 0 & X^3b &= 0, \end{aligned}$$



from which it follows that  $((X^1)^2 + (X^2)^2)a = 0$  and  $((X^3)^2 + (X^4)^2)a = 0$ , which is equivalent to

$$|Z|^2 a = 0 \quad |W|^2 b = 0.$$

Again, it follows from Proposition 6.1 that  $a = b = 0$ . This shows that  $\{E_1, E_2, E_3\}$  is basis for  $M$ .  $\square$

In the differential geometric setting, the three tangent vectors  $E_1, E_2, E_3$  are associated to the three derivations  $\partial_1, \partial_2, \partial_3$ , as given in (6.1)–(6.3). It turns out that these derivations have noncommutative analogues.

**Proposition 6.3.** *There exist hermitian derivations  $\partial_1, \partial_2, \partial_3 \in \text{Der}(S_\theta^3)$  such that*

$$\begin{aligned} \partial_1(Z) &= iZ & \partial_1(W) &= 0 \\ \partial_2(Z) &= 0 & \partial_2(W) &= iW \\ \partial_3(Z) &= Z|W|^2 & \partial_3(W) &= -W|Z|^2, \end{aligned}$$

and  $[\partial_a, \partial_b] = 0$  for  $a, b = 1, 2, 3$ .

*Proof.* Let us show that  $\partial_3$  exists; the proof that  $\partial_1, \partial_2$  exist is analogous. If  $\partial_3$  exists, the fact that it is hermitian, together with  $\partial_3(Z) = Z|W|^2$  and  $\partial_3(W) = -W|Z|^2$  completely determines  $\partial_3$  via

$$\begin{aligned} \partial_3(Z) &= Z|W|^2 & \partial_3(W) &= -W|Z|^2 \\ \partial_3(Z^*) &= Z^*|W|^2 & \partial_3(W^*) &= -W^*|Z|^2, \end{aligned}$$

since the action on an arbitrary element of  $S_\theta^3$  is given by applying Leibniz' rule repeatedly. Conversely, one may try to define  $\partial_3$  via the above relations and extend it to  $S_\theta^3$  through Leibniz' rule. However, to show that  $\partial_3$  is a derivation on  $S_\theta^3$ , one needs to check that it respects all the relations between  $Z, W$  (given in (6.5)). For instance, applying Leibniz' rule to  $\partial_3(WZ - qZW)$  gives

$$\begin{aligned} \partial_3(WZ - qZW) &= (\partial_3 W)Z + W(\partial_3 Z) - q(\partial_3 Z)W - qZ(\partial_3 W) \\ &= -W|Z|^2 Z + WZ|W|^2 - qZ|W|^2 W + qZW|Z|^2 \\ &= -(WZ - qZW)|Z|^2 + (WZ - qZW)|W|^2 = 0, \end{aligned}$$

as required (using that  $|Z|^2$  and  $|W|^2$  are central). In the same way, one may check that  $\partial_3$  is compatible with all the relations in  $S_\theta^3$  (given in (6.5)), which shows that  $\partial_3$  is indeed a derivation on  $S_\theta^3$ . To prove that  $[\partial_a, \partial_b] = 0$  one simply shows that

$$[\partial_a, \partial_b](Z) = [\partial_a, \partial_b](Z^*) = [\partial_a, \partial_b](W) = [\partial_a, \partial_b](W^*) = 0,$$

which, by Leibniz' rule, implies that  $[\partial_a, \partial_b](a) = 0$  for all  $a \in S_\theta^3$ . For instance

$$\begin{aligned} [\partial_1, \partial_3](Z) &= \partial_1(\partial_3(Z)) - \partial_3(\partial_1(Z)) = \partial_1(Z|W|^2) - \partial_3(iZ) \\ &= \partial_1(Z)|W|^2 + Z\partial_1(|W|^2) - iZ|W|^2 = Z\partial_1(WW^*) = 0. \end{aligned}$$

The remaining computations are carried out in the same manner, all giving 0.  $\square$

Next, let us construct a real metric calculus over  $S_\theta^3$ . As the metric module we choose the free module  $M$  defined in Proposition 6.2, together with the hermitian form

$$h(U, V) = \sum_{a,b=1}^3 (U^a)^* h_{ab} V^b$$

where  $U = E_a U^a$ ,  $V = E_a V^a$  and

$$(6.7) \quad (h_{ab}) = \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2 |W|^2 \end{pmatrix}.$$

(Note that  $h$  is induced from the canonical metric on the free module  $(S_\theta^3)^4$ ; i.e.  $h_{ab} = \sum_{i=1}^4 (E_a^i)^* E_b^i$ , where  $E_a = e_i E_a^i$ .) Furthermore, we let  $\mathfrak{g}$  be the (abelian) Lie algebra generated by the derivations  $\partial_1, \partial_2, \partial_3$  (in Proposition 6.3) and set  $\varphi(\partial_a) = E_a$  (and extend it as a linear map over  $\mathbb{R}$ ).

**Proposition 6.4.**  *$(M, h, \mathfrak{g}_\varphi)$  is a real metric calculus over  $S_\theta^3$ .*

*Proof.* Let us first prove that  $(M, h)$  is a metric module. From the definition of  $h$ , it is clear that  $h$  is a hermitian form, and it remains to show that it is non-degenerate. Assume that  $h(U, V) = 0$  for all  $V \in M$ . In particular, one may choose  $V = E_a$ , which gives

$$\begin{aligned} 0 &= h(U, E_1) = (U^1)^* h_{11} = (U^1)^* |Z|^2 \\ 0 &= h(U, E_2) = (U^2)^* h_{22} = (U^2)^* |W|^2 \\ 0 &= h(U, E_3) = (U^3)^* h_{33} = (U^3)^* |Z|^2 |W|^2, \end{aligned}$$

and from Proposition 6.1 it follows that  $U^1 = U^2 = U^3 = 0$ . Hence,  $h$  is non-degenerate, which shows that  $(M, h)$  is a metric module.

Moreover, it is clear that  $\varphi(\mathfrak{g})$  generates  $M$  since  $E_a = \varphi(\partial_a)$ , for  $a = 1, 2, 3$ , is in the image of  $\varphi$ . Finally, for  $E, E' \in M_\varphi$ , it is easy to see that  $h(E, E')$  is hermitian since  $h_{ab}$  is central and hermitian (cf. (6.7)).  $\square$

Since  $M$  is a free module, and  $E_a = \varphi(\partial_a)$  is a basis for  $M$ , one may use Corollary 3.8 to construct a metric and torsion-free connection on  $(M, h, \mathfrak{g}_\varphi)$ .

**Proposition 6.5.** *There exists a (unique) connection  $\nabla$  on  $(M, h, \mathfrak{g}_\varphi)$  such that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a real pseudo-Riemannian calculus. The connection is given by*

$$\begin{aligned} \nabla_1 E_1 &= -E_3 & \nabla_1 E_2 &= 0 & \nabla_1 E_3 &= E_1 |W|^2 \\ \nabla_2 E_1 &= 0 & \nabla_2 E_2 &= E_3 & \nabla_2 E_3 &= -E_2 |Z|^2 \\ \nabla_3 E_1 &= E_1 |W|^2 & \nabla_3 E_2 &= -E_2 |Z|^2 & \nabla_3 E_3 &= E_3 (|W|^2 - |Z|^2), \end{aligned}$$

where  $\nabla_a \equiv \nabla_{\partial_a}$ .

*Proof.* It is clear that  $(M, h, \mathfrak{g}_\varphi)$  satisfies the prerequisites of Corollary 3.8. Furthermore, it is a straightforward exercise to check that  $U_{ab} = \nabla_a E_b$  satisfy equation (3.11), which then implies that there exists a connection  $\nabla$  on  $(M, \mathfrak{g})$ , given by  $\nabla_a E_b$  above, such that  $(M, h, \mathfrak{g}_\varphi)$  is a pseudo-Riemannian calculus.

Let us now show that the pseudo-Riemannian calculus is real; i.e., that the elements  $h(\nabla_a \nabla_b E_p, E_q)$  are hermitian for all  $a, b, p, q \in \{1, 2, 3\}$ . We introduce the connection coefficients  $\Gamma_{ab}^c \in S_\theta^3$  through

$$\nabla_a E_b = E_c \Gamma_{ab}^c,$$

and note that  $\Gamma_{ab}^c$  is central and hermitian for all  $a, b, c \in \{1, 2, 3\}$ . It follows that

$$\begin{aligned} \nabla_a \nabla_b E_p &= \nabla_a (E_r \Gamma_{bp}^r) = (\nabla_a E_r) \Gamma_{bp}^r + E_r \partial_a \Gamma_{bp}^r \Rightarrow \\ h(\nabla_a \nabla_b E_p, E_q) &= h(\nabla_a E_r, E_q) \Gamma_{bp}^r + h(E_r, E_q) (\partial_a \Gamma_{bp}^r). \end{aligned}$$

Since  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a pseudo-Riemannian calculus and  $\Gamma_{bp}^r$  is central and hermitian, it follows that the first term is hermitian. Furthermore, since  $\partial_a$  is a hermitian derivation, and the derivative of a central element is again central, also the second term is hermitian. This shows that  $h(\nabla_a \nabla_b E_p, E_q)$  is hermitian and, hence, that  $(M, h, \mathfrak{g}_\varphi, \nabla)$  is a real pseudo-Riemannian calculus.  $\square$

Let us proceed to compute the curvature of  $(M, h, \mathfrak{g}_\varphi, \nabla)$ . Recall that since the pseudo-Riemannian calculus is real, Proposition 4.5 implies that the curvature operator has all the classical symmetries.

**Proposition 6.6.** *The curvature of the pseudo-Riemannian calculus  $(M, h, \mathfrak{g}_\varphi, \nabla)$  over  $S_\theta^3$  is given by*

$$\begin{aligned} R(\partial_1, \partial_2)E_1 &= -E_2|Z|^2 & R(\partial_1, \partial_2)E_2 &= E_1|W|^2 & R(\partial_1, \partial_2)E_3 &= 0 \\ R(\partial_1, \partial_3)E_1 &= -E_3|Z|^2 & R(\partial_1, \partial_3)E_2 &= 0 & R(\partial_1, \partial_3)E_3 &= E_1|Z|^2|W|^2 \\ R(\partial_2, \partial_3)E_1 &= 0 & R(\partial_2, \partial_3)E_2 &= -E_3|W|^2 & R(\partial_2, \partial_3)E_3 &= E_2|Z|^2|W|^2, \end{aligned}$$

from which it follows that the nonzero curvature components can be obtained from

$$R_{1212} = |Z|^2|W|^2 \quad R_{1313} = (|Z|^2)^2|W|^2 \quad R_{2323} = |Z|^2(|W|^2)^2.$$

Moreover, the (unique) scalar curvature is given by  $S = 6 \cdot \mathbf{1}$ .

*Proof.* First, it is straightforward to compute  $R(\partial_a, \partial_b)E_c$  by using the results in Proposition 6.5. For instance (recall that  $[\partial_a, \partial_b] = 0$ )

$$\begin{aligned} R(\partial_1, \partial_3)E_3 &= \nabla_1 \nabla_3 E_3 - \nabla_3 \nabla_1 E_3 = \nabla_1 (E_3(|W|^2 - |Z|^2)) - \nabla_3 (E_1|W|^2) \\ &= (\nabla_1 E_3)(|W|^2 - |Z|^2) - (\nabla_3 E_1)|W|^2 - E_1 \partial_3 |W|^2 \\ &= E_1|W|^2(|W|^2 - |Z|^2) - E_1(|W|^2)^2 - E_1(-2|W|^2|Z|^2) \\ &= E_1|Z|^2|W|^2. \end{aligned}$$

The components are easily computed as well; e.g.

$$R_{1212} = h(E_1, R(\partial_1, \partial_2)E_2) = h(E_1, E_1|W|^2) = h(E_1, E_1)|W|^2 = |Z|^2|W|^2$$

and

$$R_{1223} = h(E_1, R(\partial_2, \partial_3)E_2) = h(E_1, -E_3|W|^2) = -h(E_1, E_3)|W|^2 = 0.$$

Computing  $R_{abcd}$  for  $a, b, c, d \in \{1, 2, 3\}$  (using the symmetries in Proposition 4.5 to reduce the number of computations that need to be performed) gives  $R_{1212}$ ,  $R_{1313}$  and  $R_{2323}$  (together with the ones obtained by symmetry from these) as the only nonzero components.

Finally, let us show that there is a unique scalar curvature. The metric  $h_{ab}$  has a pseudo-inverse  $(\hat{h}^{ab}, H)$ , given by

$$(\hat{h}^{ab}) = \begin{pmatrix} |W|^2 & 0 & 0 \\ 0 & |Z|^2 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \quad \text{and} \quad H = |Z|^2|W|^2.$$

From the computation

$$\begin{aligned} \hat{h}^{ab} R_{apbq} \hat{h}^{pq} &= \hat{h}^{11} R_{1p1q} \hat{h}^{pq} + \hat{h}^{22} R_{2p2q} \hat{h}^{pq} + \hat{h}^{33} R_{3p3q} \hat{h}^{pq} = \\ &= \hat{h}^{11} (R_{1212} \hat{h}^{22} + R_{1313} \hat{h}^{33}) + \hat{h}^{22} (R_{2121} \hat{h}^{11} + R_{2323} \hat{h}^{33}) + \hat{h}^{33} (R_{3131} \hat{h}^{11} + R_{3232} \hat{h}^{22}) \\ &= 2|W|^2|Z|^2|W|^2|Z|^2 + 2|W|^2(|Z|^2)^2|W|^2 + 2|Z|^2|Z|^2(|W|^2)^2 = H(6 \cdot \mathbf{1})H, \end{aligned}$$

one concludes that the scalar curvature with respect to  $(\hat{h}^{ab}, H)$  is given by  $6 \cdot \mathbf{1}$ . Since  $H$  is central, it follows from Proposition 4.10 that this is indeed the unique scalar curvature of  $(M, h, \mathfrak{g}_\varphi, \nabla)$ .  $\square$

**6.3. Aspects of localization on  $S_\theta^3$ .** In classical geometry, the projection onto the normal space of  $S^3$  is given by the map

$$\Pi(U)^i = \Pi^{ij} U^j = x^i x^j U^j,$$

where  $x^1, \dots, x^4$  are the embedding coordinates of  $S^3$  into  $\mathbb{R}^4$ , satisfying

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1.$$

Hence, (sections of) the tangent bundle may be identified with the projective module

$$TS^3 = \mathcal{P}(C^\infty(S^3)^4)$$

where  $\mathcal{P} = \mathbf{1} - \Pi$ , giving  $TS^3$  as a subspace of  $T\mathbb{R}^4$ . It is well known that  $S^3$  is parallelizable, which means that  $TS^3$  is a free module, and one may explicitly give a basis of (global) vector fields as:

$$v_1 = (-x^4, x^3, -x^2, x^1) \quad v_2 = (-x^3, -x^4, x^1, x^2) \quad v_3 = (-x^2, x^1, x^4, -x^3).$$

The (global) vector fields  $E_1, E_2, E_3$

$$\begin{aligned} E_1 &= (-x^2, x^1, 0, 0) & E_2 &= (0, 0, -x^4, x^3) \\ E_3 &= (x^1|w|^2, x^2|w|^2, -x^3|z|^2, -x^4|z|^2) \end{aligned}$$

as defined in Section 6.1 are linearly independent at every point where  $|z|^2 = (x^1)^2 + (x^2)^2 \neq 0$  and  $|w|^2 = (x^3)^2 + (x^4)^2 \neq 0$ , which can easily be seen by computing the determinant

$$\begin{vmatrix} -x^2 & x^1 & 0 & 0 \\ 0 & 0 & -x^4 & x^3 \\ x^1|w|^2 & x^2|w|^2 & -x^3|z|^2 & -x^4|z|^2 \\ x^1 & x^2 & x^3 & x^4 \end{vmatrix} = -|z|^2|w|^2,$$

giving a condition for  $E_1, E_2, E_3, \vec{n} = (x^1, x^2, x^3, x^4)$  to be linearly independent. Thus, the vector fields  $E_1, E_2, E_3$  provide a globalization of the corresponding vector fields in the local chart defined by the Hopf coordinates, and one may use them for computations, keeping in mind that they do not span the tangent space at points  $(x^1, x^2, x^3, x^4) \in S^3$  where  $x^1 = x^2 = 0$  or  $x^3 = x^4 = 0$ . However, in this case, the set of points on  $S^3$  which are not covered by this chart has measure zero, which implies that certain results, e.g. results involving integration over the manifold, is not sensitive to the difference between  $\{E_1, E_2, E_3\}$  and  $\{v_1, v_2, v_3\}$ .

Returning to the noncommutative 3-sphere  $S_\theta^3$ , it is easy to check that since

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = ZZ^* + WW^* = \mathbf{1}$$

the above situation allows for a straightforward generalization. Namely, for  $U = e_i U^i \in (S_\theta^3)^4$  one defines the map  $\mathcal{P} : (S_\theta^3)^4 \rightarrow (S_\theta^3)^4$  as

$$\mathcal{P}(U) = \sum_{i,j=1}^4 e_i \mathcal{P}^{ij} U^j$$

where  $\mathcal{P}^{ij} = \delta^{ij}\mathbb{1} - X^i X^j$ , and it is easy to check that  $\mathcal{P}^2(U) = \mathcal{P}(U)$ . Hence,  $TS_\theta^3 = \mathcal{P}((S_\theta^3)^4)$  is a projective module in close analogy with the module of vector fields on  $S^3$ . Let us now further study the structure of  $TS_\theta^3$ . We start by proving the following lemma.

**Lemma 6.7.** *In  $S_\theta^3$*

$$(6.8) \quad X^2 X^4 + X^1 X^3 = q(X^4 X^2 + X^3 X^1)$$

$$(6.9) \quad X^2 X^4 - X^1 X^3 = \bar{q}(X^4 X^2 - X^3 X^1)$$

$$(6.10) \quad X^2 X^3 + X^1 X^4 = \bar{q}(X^3 X^2 + X^4 X^1)$$

$$(6.11) \quad X^2 X^3 - X^1 X^4 = q(X^3 X^2 - X^4 X^1).$$

*Proof.* The proof is done by straightforward computations; e.g.

$$\begin{aligned} X^2 X^3 + X^1 X^4 &= \frac{1}{2i}(Z - Z^*)\frac{1}{2}(W + W^*) + \frac{1}{2}(Z + Z^*)\frac{1}{2i}(W - W^*) \\ &= \frac{1}{2i}(ZW - Z^*W^*) = \frac{\bar{q}}{2i}(WZ - W^*Z^*) \\ &= \frac{1}{2}(W + W^*)\frac{1}{2i}(Z - Z^*) + \frac{1}{2i}(W - W^*)\frac{1}{2}(Z + Z^*) \\ &= \bar{q}(X^3 X^2 + X^4 X^1), \end{aligned}$$

and the remaining computations are completely analogous.  $\square$

The next statement corresponds to the fact that  $S^3$  is a parallelizable manifold.

**Proposition 6.8.** *The (right)  $S_\theta^3$ -module  $TS_\theta^3$  is a free module with basis*

$$\begin{aligned} F_1 &= (-X^4, X^3, -qX^2, qX^1) \\ F_2 &= (-X^3, -X^4, qX^1, qX^2) \\ F_3 &= (-X^2, X^1, X^4, -X^3). \end{aligned}$$

*Proof.* Let us start by showing that  $\Pi(F_a) = 0$ , which implies that  $F_a \in TS_\theta^3$ . Since  $\Pi^{ij} = X^i X^j$ , it is enough to show that  $X^i F_a^i = 0$  for  $a = 1, 2, 3$ :

$$\begin{aligned} X^i F_1^i &= -X^1 X^4 + X^2 X^3 - qX^3 X^2 + qX^4 X^1 = 0 \\ X^i F_2^i &= -X^1 X^3 - X^2 X^4 + qX^3 X^1 + qX^4 X^2 = 0 \\ X^i F_3^i &= -X^1 X^2 + X^2 X^1 + X^3 X^4 - X^4 X^3 = 0, \end{aligned}$$

by using (6.11), (6.8) in Lemma 6.7, and the fact that  $[X^1, X^2] = [X^3, X^4] = 0$ .

Next, we show that  $F_1, F_2, F_3$  generate  $TS_\theta^3$ ; it is sufficient to show that  $\mathcal{P}(e_i)$  (where  $\{e_i\}_{i=1}^4$  denotes the canonical basis of  $(S_\theta^3)^4$ ) can be written as linear combination of  $F_1, F_2, F_3$ , for  $i = 1, 2, 3, 4$ . In fact, one can show that

$$\begin{aligned} \mathcal{P}(e_1) &= (\mathbb{1} - (X^1)^2, -X^2 X^1, -X^3 X^1, -X^4 X^1) = -F_1 X^4 - F_2 X^3 - F_3 X^2 \\ \mathcal{P}(e_2) &= (-X^1 X^2, \mathbb{1} - (X^2)^2, -X^3 X^2, -X^4 X^2) = F_1 X^3 - F_2 X^4 + F_3 X^1 \\ \mathcal{P}(e_3) &= (-X^1 X^3, -X^2 X^3, \mathbb{1} - (X^3)^2, X^4 X^3) = -\bar{q}F_1 X^2 + \bar{q}F_2 X^1 + F_3 X^4 \\ \mathcal{P}(e_4) &= (-X^1 X^4, -X^2 X^4, -X^3 X^4, \mathbb{1} - (X^4)^2) = \bar{q}F_1 X^1 + \bar{q}F_2 X^2 - F_3 X^3. \end{aligned}$$

For instance,

$$\begin{aligned} -F_1X^4 - F_2X^3 - F_3X^2 &= ((X^2)^2 + (X^3)^2 + (X^4)^2, -X^3X^4 + X^4X^3 - X^1X^2, \\ &\quad qX^2X^4 - qX^1X^3 - X^4X^2, -qX^1X^4 - qX^2X^3 + X^3X^2) \\ &= (\mathbb{1} - (X^1)^2, -X^2X^1, -X^3X^1, -X^4X^1) = \mathcal{P}(e_1), \end{aligned}$$

by using (6.9), (6.10) (in the third and fourth component, respectively) and the fact that  $[X^1, X^2] = [X^3, X^4] = 0$ . Finally, let us show that  $F_1, F_2, F_3$  are free generators. For  $a, b, c \in S_\theta^3$ , we assume that

$$F_1a + F_2b + F_3c = 0,$$

which is equivalent to

$$\begin{cases} -X^4a - X^3b - X^2c = 0 \\ X^3a - X^4b + X^1c = 0 \\ -qX^2a + qX^1b + X^4c = 0 \\ qX^1a + qX^2b - X^3c = 0. \end{cases}$$

Multiplying these equations (from the left) by  $-X^2, X^1, X^4$  and  $-X^3$ , respectively, and summing them yields  $c = 0$ , by using (6.8) and (6.11). Setting  $c = 0$  in the above equations gives

$$\begin{aligned} X^4a + X^3b &= 0 & X^3a - X^4b &= 0 \\ -X^2a + X^1b &= 0 & X^1a + X^2b &= 0, \end{aligned}$$

which implies that

$$\begin{aligned} (X^4)^2a &= -X^4X^3b & (X^3)^2a &= X^3X^4b \\ (X^2)^2a &= X^2X^1b & (X^1)^2a &= -X^1X^2b. \end{aligned}$$

Summing these equations gives  $a = 0$ , which then (via a similar argument) implies that  $b = 0$ . This shows that  $F_1, F_2, F_3$  are linearly independent.  $\square$

It is easy to check that the elements  $E_1, E_2, E_3$ , as defined in (6.6), fulfill  $\mathcal{P}(E_a) = E_a$  for  $a = 1, 2, 3$ , implying that they are elements of  $TS_\theta^3$ . Hence, the module  $M$ , of the pseudo-Riemannian calculus for  $S_\theta^3$ , is a submodule of  $TS_\theta^3$ , providing a noncommutative analogue of the globalization of the local vector fields in the Hopf coordinates as described in the beginning of the section.

As is well known, every projective module comes equipped with a canonical affine connection; namely, the module  $(S_\theta^3)^4$  has an affine connection, given by

$$\bar{\nabla}_d V = e_i d(V^i)$$

where  $V = e_i V^i \in (S_\theta^3)^4$  and  $d \in \text{Der}(S_\theta^3)$ , and it follows that

$$\hat{\nabla}_d V = \mathcal{P}(\bar{\nabla}_d V)$$

is an affine connection on  $TS_\theta^3$ . Since we have argued in analogy with differential geometry, where  $M$  is a sub-module of  $TS^3$  and the connection on  $M$  is merely the restriction of the connection on  $TS^3$ , it is natural to ask if the connection  $\hat{\nabla}$  (restricted to  $M$ ) coincides with  $\nabla$  (as given by the pseudo-Riemannian calculus over  $M$ ).

**Proposition 6.9.** *Let  $(M, h, \mathfrak{g}_\varphi, \nabla)$  be the pseudo-Riemannian calculus over  $S_\theta^3$  introduced in Section 6.2. The affine connection  $\hat{\nabla}_d U = \mathcal{P}(\bar{\nabla}_d U)$ , restricted to  $M \subseteq TS_\theta^3$ , coincides with  $\nabla$ ; i.e.,  $\hat{\nabla}_d U = \nabla_d U$  for  $d \in \mathfrak{g}$  and  $U \in M$ .*

*Proof.* The proof is easily done by a straightforward computation, where one computes  $\hat{\nabla}_a E_b$  for  $a, b = 1, 2, 3$ , and compares it with the result in Proposition 6.5. For instance,

$$\begin{aligned} \hat{\nabla}_1 E_1 &= \mathcal{P}((-\partial_1 X^2, \partial_1 X^1, 0, 0)) = \mathcal{P}((-X^1, -X^2, 0, 0)) \\ &= (-X^1, -X^2, 0, 0) - (X^1, X^2, X^3, X^4)((-X^1)^2 - (X^2)^2) \\ &= (X^1(|Z|^2 - \mathbb{1}), X^2(|Z|^2 - \mathbb{1}), X^3|Z|^2, X^4|Z|^2) \\ &= (-X^1|W|^2, -X^2|W|^2, X^3|Z|^2, X^4|Z|^2) = -E_3, \end{aligned}$$

which coincides with  $\nabla_1 E_1$ .  $\square$

In order to take the analogy with localization one step further, let us introduce a localized algebra  $S_{\theta, \text{loc}}^3$  constructed by formally adjoining the inverses of  $|Z|^2$  and  $|W|^2$  to the algebra  $S_\theta^3$ . More precisely, the multiplicative set  $S$  generated by  $|Z|^2, |W|^2, \mathbb{1}$  trivially satisfies the (right and left) Ore condition (since it consists of central elements) and the fact that  $|Z|^2, |W|^2$  are regular elements (cf. Proposition 6.1) implies that the Ore localization at  $S$  exists (see e.g. [Coh95]). If we consider  $TS_\theta^3$  and  $M$  as (right)  $S_{\theta, \text{loc}}^3$ -modules, they coincide, which we show by explicitly finding a relation between the two sets of generators.

**Proposition 6.10.** *Consider the following elements of  $(S_{\theta, \text{loc}}^3)^4$ :*

$$\begin{aligned} F_1 &= (-X^4, X^3, -qX^2, qX^1) & E_1 &= (-X^2, X^1, 0, 0) \\ F_2 &= (-X^3, -X^4, qX^1, qX^2) & E_2 &= (0, 0, -X^4, X^3) \\ F_3 &= (-X^2, X^1, X^4, -X^3) & E_3 &= (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2). \end{aligned}$$

*Then it holds that*

$$\begin{aligned} F_1 &= E_1|Z|^{-2}(X^1X^3 + X^2X^4) + E_2|W|^{-2}(X^1X^3 + X^2X^4) \\ &\quad + E_3|Z|^{-2}|W|^{-2}(X^2X^3 - X^1X^4) \\ F_2 &= E_1|Z|^{-2}(X^2X^3 - X^1X^4) + E_2|W|^{-2}(X^2X^3 - X^1X^4) \\ &\quad - E_3|Z|^{-2}|W|^{-2}(X^1X^3 + X^2X^4) \\ F_3 &= E_1 - E_2. \end{aligned}$$

*Proof.* Let us show that  $F_1$  can be written as a linear combination of  $E_1, E_2, E_3$ , as given in the statement. Namely, introducing  $W^i$  through

$$\begin{aligned} e_i W^i &= E_1|Z|^{-2}(X^1X^3 + X^2X^4) + E_2|W|^{-2}(X^1X^3 + X^2X^4) \\ &\quad + E_3|Z|^{-2}|W|^{-2}(X^2X^3 - X^1X^4), \end{aligned}$$

gives

$$\begin{aligned} W^1 &= -X^2|Z|^{-2}(X^1X^3 + X^2X^4) + X^1|Z|^{-2}(X^2X^3 - X^1X^4) \\ W^2 &= X^1|Z|^{-2}(X^1X^3 + X^2X^4) + X^2|Z|^{-2}(X^2X^3 - X^1X^4) \\ W^3 &= -X^4|W|^{-2}(X^1X^3 + X^2X^4) - X^3|W|^{-2}(X^2X^3 - X^1X^4) \\ W^4 &= X^3|W|^{-2}(X^1X^3 + X^2X^4) - X^4|W|^{-2}(X^2X^3 - X^1X^4). \end{aligned}$$

Using the fact that  $[X^1, X^2] = 0$  (in  $W^1, W^2$ ), together with (6.8) and (6.11) (in  $W^3, W^4$ ), yields

$$\begin{aligned} W^1 &= -|Z|^{-2}((X^2)^2 + (X^1)^2)X^4 = -|Z|^{-2}|Z|^2X^4 = -X^4 \\ W^2 &= |Z|^{-2}((X^1)^2 + (X^2)^2)X^3 = |Z|^{-2}|Z|^2X^3 = X^3 \\ W^3 &= -q|W|^{-2}((X^4)^2 + (X^3)^2)X^2 = -q|W|^{-2}|W|^2X^2 = -qX^2 \\ W^4 &= q|W|^{-2}((X^4)^2 + (X^3)^2)X^1 = q|W|^{-2}|W|^2X^1 = qX^1, \end{aligned}$$

which shows that  $e_i W^i = F_1$ .  $\square$

Finally, we note that the metric

$$(h_{ab}) = \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2|W|^2 \end{pmatrix}$$

is invertible in  $S_{\theta, \text{loc}}^3$ , giving a local calculus in almost complete analogy with differential geometry.

#### ACKNOWLEDGMENT

We would like to thank L. Dabrowski, M. Khalkhali, G. Landi, and A. Sitarz for interesting and useful discussions during the HIM trimester program “Noncommutative Geometry and its Applications” in the fall of 2014, as well as the University of Western Ontario for hospitality. Furthermore, J.A. is supported by the Swedish Research Council.

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